MPSI Class Entrance Test 2005

Test time : 4 hours
English Version

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The following exercises can be solved independently and done in any order. The question are listed from easiest to most difficult. Calculators are not permitted. Solutions should be written in French.

1. Let ABC be a triangle and C' the midpoint of $[AB]$. Compare $AC + BC$ and CC'.

2. Knowing that $\sqrt{30} \notin \mathbb{Q}$, show that $\sqrt{2} + \sqrt{3} + \sqrt{5} \notin \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers.

3. The integer part of a real number x, denoted by $E(x)$, is the largest integer that is less or equal to x. For example, $E(\pi) = 3$, $E(-\pi) = -4$. Solve the equation $E(2x+1) = E(x+4)$.

4. Let $n \geqslant 3$ be an integer and set $E = \{1, 2, \dots, n\}$.
   (a) Find the number of ways to choose a pair $\{a, b\}$ of distinct and non consecutive integers from the set E.
   (b) Let $p \leqslant n/2$ be an integer. Find the number of subset of E of cardinal p that contains no consecutive integers.

5. Let $n \geqslant 2$ be an integer. We note $\mathrm{d}(n)$ the number of divisors of n (in $\mathbb{N}^*$).
   (a) We write the factorization of n as a product of primes :
   $$n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_r}^{\alpha_r}$$
   where $p_1 < p_2 < \cdots < p_r$ are prime numbers and $\alpha_1, \dots, \alpha_r$ integers. Calculate $\mathrm{d}(n)$ in terms of $\alpha_1, \dots, \alpha_r$.
   (b) When is $\mathrm{d}(n)$ odd ?
   (c) The sequence $(n_k)_{k \geqslant 0}$ is defined by :
   $$n_0 = n \qquad \text{and} \qquad \forall k \in \mathbb{N}, \quad n_{k+1} = \mathrm{d}(n_k).$$
   Find all the integers n such that the set $\{n_k,\ k \in \mathbb{N}\}$ contains no square.

6. Let $n \geqslant 1$ be an integer. Suppose that we can find $2k$ distinct positive integers such that the sums $a_1 + b_1, \dots, a_k + b_k$ are all distinct and strictly less than n. Prove that $k \leqslant \dfrac{2n-3}{5}$.

7. The plane is painted with two colors. Show that we can always find an isosceles right triangle whose vertex have the same color.

8. Let $n \geqslant 1$ be an integer. Find all real numbers x such that $(\cos x)^n - (\sin x)^n = 1$.

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