MPSI Class Entrance Test
2006

Test time : 4 hours
English Version

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The following exercises can be solved independently and done in any order. The question are listed from easiest to most difficult. Calculators are not permitted. Solutions should be written in French or in English.

1. Let x be a real number that satisfies $x^3 + \dfrac{1}{x^3} = 2\sqrt{5}$. Find the value of $x^2 + \dfrac{1}{x^2}$.

2. Let (A_1, A_2, A_3) be a triangle of the plane. Let A_4 be the orthocentre of the triangle and G the center of gravity of A_1, A_2, A_3, A_4. For $1 \leqslant i < j \leqslant 4$ let $A_{i,j}$ be the midpoint of $[A_iA_j]$.

Show that the six distances $A_{i,j}G$ (for $1 \leqslant i < j \leqslant 4$) are the same.

3. Let a, b, c be the side's length of a triangle ABC.

(a) Prove that we can find three strictly positive real numbers x, y, z such that $a = x + y$, $b = y + z$ and $c = z + x$.

(b) Show that $\dfrac{3}{2} \leqslant \dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b} < 2$.

4. Let f be a map from the plane $\mathbb{R}^2$ into itself. For $d \in \mathbb{R}_+$ we say that f preserves the distance d if we have $AB = d \Longrightarrow f(A)f(B) = d$ for every pair of points (A, B). We suppose that f preserves the distance 1 and is injective wich means that $A \neq B \Longrightarrow f(A) \neq f(B)$.

(a) Prove that f preserves the distance 2 and then every integral distance.

(b) Prove that f preserves the distance $\sqrt{3}$.

5. Let n and k be positive integers such that $1 \leqslant k \leqslant n - 1$.

When is the binomial coefficient $\dbinom{n}{k}$ prime ?

Remember that an integer $m \geqslant 2$ is called prime if the only positive divisors of m are 1 and m.

6. If m is an integer, we note $f(m)$ the biggest odd divisor of m. For $n \geqslant 1$, evaluate $\displaystyle\sum_{k=1}^{2^n} f(k)$.

7. Let ε be in $]0, 1[$ and n in $\mathbb{N}^*$. We consider n positive real numbers $x_1, \ldots, x_n$ such that :

$$\forall (i,j) \in \{1, \ldots, n\}^2, \qquad x_i x_j \leqslant \varepsilon^{|i-j|}.$$

Show that :

$$\sum_{i=1}^{n} x_i \leqslant \frac{1}{1 - \sqrt{\varepsilon}}\,.$$

8. For $n \geqslant 2$, we note p_n the number of permutations σ of the set $\{1, \ldots, n\}$ such that there is a single integer $i \in \{1, 2, \ldots, n-1\}$ such that $\sigma(i) > \sigma(i+1)$. Find p_n.

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