MPSI Class Entrance Test
2006

Test time : 4 hours
English Version

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The following exercises can be solved independently and done in any order. The question are listed from easiest to most difficult. Calculators are not permitted. Solutions should be written in French or in English.

1. Calculate :

$$\cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right).$$

2. For $n \in \mathbb{N}^*$ we note $n! = 1 \times 2 \times \cdots \times n$.

Find all quadruples (x, y, z, t) of strictly positive integers such that $x! + y! + z! = t!$.

3. The integer part of a real number x, denoted by $E(x)$, is the largest integer that is less or equal to x.

Find all integers $n \geqslant 1$ such that $E\left(\sqrt{n}\right)$ divides n.

4. Let V be the volume of a cuboid (this is a rectangular parallelepiped) and S the total surface area.

What is the maximal value of $\dfrac{V^2}{S^3}$?

5. Let a, b, c be three positive real numbers.

Prove that at least one of the numbers $a(1-b)$, $b(1-c)$, $c(1-a)$ is lower or equal than $\dfrac{1}{4}$.

6. Show that an equilateral triangle cannot have all his three vertices with integral coordinates.

7. If M and N are two points of the plane we note MN the distance between M and N. If (A, B, C) is a triple of points let $p(A, B, C) = AB + AC + BC$. Let f be a function from the plane into itself such that $p(f(A), f(B), f(C)) = p(A, B, C)$ for all triple (A, B, C). Show that f is an isometry.

8. Let a and b be integers such that $2 \leqslant a \leqslant b$. For all $n \geqslant 1$ let $u_n = \dfrac{b^n - 1}{a^n - 1}$.

(a) Suppose that there is an integer $k \in \mathbb{N}^*$ such that $b = a^k$. Prove that $u_n \in \mathbb{N}^*$ for all $n \in \mathbb{N}^*$.

(b) Suppose now that $a < b < a^2$. By considering $au_{n+1} - bu_n$, show that we can find $n \in \mathbb{N}^*$ such that $u_n \notin \mathbb{N}^*$.

(c) Suppose now that $a^2 < b < a^3$. Show that we can find $n \in \mathbb{N}^*$ such that $u_n \notin \mathbb{N}^*$.

(d) How can we extend the result of questions (b) and (c) ?

9. Let $(m, n) \in (\mathbb{N}^*)^2$. Find the number of sequences $(x_1, \ldots, x_n)$ of $\{0,1\}^n$ such that :

$$\{i \in \{1, \ldots, n-1\},\ x_i = 0 \text{ et } x_{i+1} = 1\}$$

has cardinal m ?

It is asked to give the result as a binomial coefficient $\dbinom{a}{b}$ where (a, b) depends on (n, m).

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