French MPSI Class
Entrance Test
2007

Test time : 4 hours
English Version

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The following exercises can be solved independently and done in any order. The question are listed from easiest to most difficult. Calculators are not permitted. Solutions should be written in French or in English.

1. Find all sequences $(a_n)_{n\geqslant 1}$ of strictly positive real numbers such that :

$$\forall n \in \mathbb{N}, \quad \sum_{i=1}^{n} a_i^3 = \left(\sum_{i=1}^{n} a_i\right)^2$$

2. Let ABC be a triangle and note A' the midpoint of $[BC]$, B' the midpoint of $[AC]$ and C' the midpoint of $[AB]$. Find the value of :

$$\frac{AA'^2 + BB'^2 + CC'^2}{BC^2 + AC^2 + AB^2}$$

3. A 100 meters long racing cyclists group is riding at constant speed v. A motorcycle, riding at constant speed V, starts form the last cyclist, joins the first one, turns instantaneously back and returns to the last cyclist. During this time the group covered 100 meters. Find the distance covered by the motorcycle.

4. Let a, b, c be three positive real numbers such that $abc = 1$. Show that

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geqslant \frac{3}{4}$$

5. Let $f : [0,1] \to \mathbb{R}$ be a function that satisfies :

(i) $\quad f(1) = 1$

(ii) $\quad \forall x \in [0,1], \quad f(x) \geqslant 0$

(iii) $\quad \forall (x,y) \in [0,1]^2, \quad x + y \leqslant 1 \Longrightarrow f(x) + f(y) \leqslant f(x+y)$.

Prove that for all $x \in [0,1]$, $f(x) \leqslant 2x$.

6. Remember that if $m \geqslant 1$ is an integer, we set $m! = 1 \times 2 \times 3 \cdots \times m$.

(a) Find all integers $n \geqslant 2$ such that n divides $(n-1)!$.

(b) Let $n \geqslant 2$ and $k \geqslant 1$ be such that $n-1$ divides $1 + n + n^2 + \cdots + n^{k-1}$.
Prove that $n-1$ divides k.

(c) Find all integers $n \geqslant 2$ such that $(n-1)! + 1$ is a power of n.

7. Find the maximum possible product of a set of positive integers with sum 2006.

8. Let $n = 2^p$ be a power of 2. We consider the subsets A of the set $E = \{1, 2, ..., n\}$ having the following property : if $x \in A$ then $2x \notin A$. How many integers at most can such a subset A contain ?

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