MPSI Class Entrance Test 2008
Test time : 4 hours
English Version

The following exercises can be solved independently and done in any order. The question are listed from ɑsiest to most difficult. Calculators are not permitted. Solutions should be written in french or in english.

1. Prove that for any triangle with sides a, b, c and area A we have

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}A$$

When does equality occur ?

2. Let $p \geqslant 3$ be a prime number. Prove that, for any $n \geqslant 1$, the number $u_n = \sum_{k=1}^{n} \frac{p^k}{k}$ can be written $u_n = p\frac{a_n}{b_n}$ where a_n et b_n are integers that can not be divided by p.

3. Let $(u_n)_{n\geqslant 1}$ be the sequence defined by $u_1 = 1$, $u_2 = u_3 = 2$, $u_4 = u_5 = u_6 = 3$, $u_7 = u_8 = u_9 = u_{10} = 4, \ldots$ Prove that for any $n \geqslant 1$ we have

$$u_n = \left[\frac{1+\sqrt{8n-7}}{2}\right]$$

We note $[x]$ the integer part of the real number x. For example $[\pi] = 3$.

4. Let T be the set of polynomial functions that can be written

$$x \mapsto x^2 + ax + b, \quad (a, b) \in \mathbb{Z}^2$$

Let E be the set of real numbers $x > 1$ for which we can find f in T and y in $]-1, 1[$ such that $f(x) = f(y) = 0$.

(a) Prove that the number $k + \sqrt{k^2 - 1}$ is in E for any integer $k \geqslant 2$.

(b) If $M > 1$ prove that the set $E \cap [1, M]$ is finite.

(c) What is the smallest number of E ?

5. Find the number of ways to place n undistinguishable rooks on an $n \times n$ chessboard such that no rook can attack another one, and such that the placing is invariant with respect to a quarter-turn of the board.

Remember that two rooks can attack each other when they are on the same line or on the same column.

6. Let P be a full convex polygon of the plane. For $r \geqslant 0$ we note $A(r)$ the area of the set of points whose distance to P is $\leqslant r$. Find the expression of $A(r)$ as a function of r, of the area $A(0) = A$ of P and of the perimeter ℓ of P.

7. Let S be a non-empty subset of $\mathbb{R}^3$ such that for any plane P which cuts S the intersection $S \cap P$ is a circle (or a point) of the plane P. What can we say about S ?

8. Find the smallest integer $m \geqslant 1$ such that we can find m points of the plane $A_1, A_2, \ldots, A_m$ which have the following property : for any point M of the plane, at least one of the distances A_kM is irrational.