MPSI Class Entrance Test 2010

Test time：4 hours

English Version

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The following exercises can be solved independently and done in any order. The question are listed from easiest to most difficult. Calculators are not permitted. Solutions should be written in French or in English.

1. Compare the numbers π^e and e^π.
2. Three circles of radius 1 meet in a point *O* and intersect again pairwise in *A*, *B*, *C*.
   Prove that *O* is the orthocenter of the triangle (*A*, *B*, *C*).
3. Let *M* be a point in the interior (strictly) of a *n*-regular polygon with sides of length *a*.
   Let $d_1,\cdots,d_n$ be the distance between Mand the different sides of the polygon.
   Prove that
   $$\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_n}>\frac{2\pi}{a}$$
4. Let $f:R\to R$ be a function and note *E* the set containing all centers of symmetry of *f*'s graph.
   (a) Find *E* when *f* is the map $x\mapsto x+\sin x$.
   (b) We suppose that *E* contains at least two points. Prove that *f* can be written as the sum of an affine function and a periodic function.
   (c) Is-it possible for *E* to contain 3 unaligned points ?
5. Let *ABC* be a equilateral triangle of a plane *P*, *O* the center of *ABC*, Δ the line perpendicular to *P* which passes through *O*, *D* a point on Δ with $D\neq O$. Express the radius of the sphere containing *A,B,C,D* as a function of $l=AB$ and $e=OD$.
6. Let *X* be a non-empty subset of N^* such that : for all $x\in X$, $4x\in X$ and $[\sqrt{x}]\in X$.
   Prove that $X=N^*$. *For* $y\in R$, $[y]$ *is the integer part of* y.
7. Let *S* be a sphere of radius 1 and center *O* in the Euclidean space. For *A,B,C* on *S* we consider
   $$f(A,B,C)=\left\langle\overrightarrow{OA},\overrightarrow{OB}\right\rangle+\left\langle\overrightarrow{OB},\overrightarrow{OC}\right\rangle+\left\langle\overrightarrow{OC},\overrightarrow{OA}\right\rangle$$
   find the maximal and minimal value of f.
8. We consider an alphabet with n letters. We note *M* the set of words with the following property : when two letters of the word are the same, all the letters between them are different. For example, when the alphabet is $\{a,b,c\}$ the word *abbacb* is not in M but the word *bbacbac* is in *M*.
   (a) What is the maximal length of a word in *M* ?
   (b) How many words of maximal lengths are in *M* ?