A Brief Correction of 2015 Session

The answer may be wrong or have some errors.

Qusetion 1:
Let $a_1, ..., a_n$ be strictly positive real numbers, σ a permutation of $\{1, .., n\}$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{a_{\sigma(i)}} \geq n$$

A. Without loss of generality, one may assume that $a_1 \geq ... \geq a_n$. If $n = 2$, one can easily verify the inequality. Suppose that the inequality is verified for all $n < m$. For the case $n = m$, we distinguish two subcases:
(1) $\sigma(n) = n$. In this subcase we may directly apply the hypothesis of induction to conclude.
(2) $\sigma(n) = i < n$. Let j such that $\sigma(j) = n$. In this case we have

$$\frac{a_n}{a_i} + \frac{a_j}{a_n} \geq \frac{a_n}{a_n} + \frac{a_j}{a_i}.$$

We define $\phi(k) = \sigma(k)$ if $k \neq j, n$ and $\phi(n) = n, \phi(j) = i$. Then we have

$$\sum_{i=1}^{n} \frac{a_i}{a_{\sigma(i)}} \geq \sum_{i=1}^{n} \frac{a_i}{a_{\phi(i)}} \geq n$$

by induction's hypothesis, which ends the induction.

A. Another way is to observe that

$$\ln(\frac{1}{n}\sum_{i=1}^{n} \frac{a_i}{a_{\sigma(i)}}) \leq \frac{1}{n}\sum_{i=1}^{n} \ln(\frac{a_i}{a_{\sigma(i)}}) = 0.$$

and therefore

$$\sum_{i=1}^{n} \frac{a_i}{a_{\sigma(i)}} \geq n.$$

Qusetion 2:
Let C_1 and C_2 be two circles, that intersect at two points A and B. Let δ be a straight line, tangent to both circles respectively at two points M and N. Prove that the line (AB) intersects the line segment $[MN]$ at its midpoint.

A. Let O_1 and O_2 the center of each circle, H the intersection of (AB) and $[MN]$ and P the intersection of $[AB]$ and $[O_1O_2]$. Then one has

$$\begin{aligned} MH^2 &= HP^2 + O_1P^2 - O_1M^2 \\ &= HP^2 - AP \cdot PB \\ &= HP^2 + O_2P^2 - O_2N^2 \\ &= NH^2 \end{aligned}$$

and therefore H is the midpoint of $[NM]$.

Qusetion 3:
Prove that the inequality $\sin(\cos x) < \cos(\sin x)$ holds for every real number x.

A. We notice that $\cos(\sin x) > 0$ and each term is even, so we only need to study the case $x \in]0, \frac{\pi}{2}[$. Or one has $\forall x \in]0, \frac{\pi}{2}[, x > \sin x$, so

$$\sin(\cos x) < \cos x < \cos(\sin x).$$

Qusetion 4:
Find all three-digit integers n such that the decimal expansion for n^2 terminates with n.

A. Let $100a + 10b + c$ the decimal representation of n. Then it is easy to see that $c \in \{0, 1, 5, 6\}$. Besides, $n^2 \equiv 100b^2 + c + 2 + 20bc + 200ac \equiv 100a + 10b + c \mod 1000$ and a simple discussion gives us $n = 625, 376$.

Qusetion 5:
Let $(u_n)_{n\geq}0$ a sequence of non-negative real numbers such that $u_0 = 1$. Assume moreover that, for every integer $n \geq 1$, at least half of terms $u_0, ..., u_{n-1}$ are bigger than or equal to $2u_n$. Prove that $(u_n)_{n\geq}0$ converge to 0.

A. It is easy to see that $\forall i, u_i \leq u_0 = 1$. Therefore, the sequence $\forall i, v_i = \sup_{k\geq i} u_k$ is well defined. one can see that this sequence (v_i) is decreasing and "greater" than (u_i): $\forall i, v_i \geq v_{i+1}$ and $v_i \geq u_i$. Let (w_i) be a sequence defined by $w_0 = 1, \forall i, w_{i+1} = \frac{1}{2} w_{\frac{i-1}{2}}$ if i is odd and $\frac{1}{2} w_{\frac{i}{2}}$ if not. It is not difficult to prove by induction that (w_i) is decreasing, greater than (v_i) and $\forall i \geq 1, w_{2^i} = \frac{1}{2^i}$. Since (w_i) converge to zero, (u_n) converge to zero.

Qusetion 6:
(a) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every $y \in \mathbb{R}$, the equation $f(x) = y$ has exactly three solutions.
(b) Is it possible to find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every $y \in \mathbb{R}$, the equation $f(x) = y$ has exactly two solutions?

A. (a) Let

$$\begin{aligned} f(x) &= (x - 3k) + k \qquad \forall x \in [3k, 3k + 1] \\ &= -(x - 3k) + 2 + k \; \forall x \in [3k + 1, 3k + 2] \\ &= (x - 3k) - 2 + k \quad \forall x \in [3k + 2, 3k + 3] \end{aligned}$$

It is easy to see that this function answers to the question.
(b) It is impossible. Suppose that such a function f exists. Let $a \neq b$ be solutions of $f(x) = 0$. Without loss of generality we may assume that $\forall x \in]a, b[, f(x) > 0$. Let $\epsilon > 0$ be close enough to zero. Then $f(x) = \epsilon$ has exactly two solutions (one around a and the other one around b). However this is contradictory since for a great enough y there is no solution between $]a, b[$ for $f(x) = y$. Therefore there must be a solution $< a$ or $> b$ for this equation, but then $f(x) = \epsilon$ has another distinct solution, which is once moreover contradictory. Thus, such a function doesn't exist.

Qusetion 7:
A balanced coin is tossed n times, and we let F be the number of "heads" and P the number of "tails" obtained.

(a) Prove that the expectation (i.e the average) of $|F-P|$ equals

$$\frac{1}{2^{n-1}}\sum_{k<\frac{n}{2}}(n-2k)\binom{n}{k}$$

(b) Find a simple expression for the value of this sum whenever $n=2p$.

A. (a)One has

$$\begin{aligned}\mathbb{E}(|F-P|) &= \sum_{k=0}^{n}|k-(n-k)|\mathbb{P}(F=k,P=n-k)\\ &= \sum_{k<\frac{n}{2}}(n-2k)(\mathbb{P}(F=k,P=n-k)+\mathbb{P}(F=n-k,P=k))\\ &= \sum_{k<\frac{n}{2}}(n-2k)\frac{1}{2^{n-1}}\binom{n}{k}\end{aligned}$$

(b) If $n=2$, then $\mathbb{E}(|F-P|)=1$. If $n=2p$ with $p\geq 2$, then

$$\begin{aligned}\mathbb{E}(|F-P|) &= \sum_{k<\frac{n}{2}}(n-2k)\frac{1}{2^{n-1}}\binom{n}{k}\\ &= \sum_{k<\frac{n}{2}}(2n-2k)\frac{1}{2^{n-1}}\binom{n}{k}-\sum_{k<\frac{n}{2}}(n)\frac{1}{2^{n-1}}\binom{n}{k}\\ &= \sum_{1\leq k<\frac{n}{2}}(n-k)\frac{2}{2^{n-1}}(n)\binom{n-1}{k}-\sum_{1\leq k<\frac{n}{2}}(n)\frac{1}{2^{n-1}}(\binom{n-1}{k}+\binom{n-1}{k-1})+\frac{n}{2^{n-1}}\\ &= \sum_{1\leq k<\frac{n}{2}}(n-k)\frac{2}{2^{n-1}}(n)(\binom{n-1}{k}-\binom{n-1}{k-1})+\frac{n}{2^{n-1}}\\ &= \frac{n}{2^{n-1}}(\binom{n-1}{\frac{n}{2}-1}-1+1)\\ &= \frac{n}{2^{n-1}}\binom{n-1}{\frac{n}{2}-1}\end{aligned}$$

Qusetion 8:

Find all non-decreasing functions $f:\mathbb{N}^*\to\mathbb{R}_+^*$ such that

$$\forall(k,m)\in(\mathbb{N}^*)^2, f(m^k)=f(m)^k$$

A. Immediately we see that $f(1)=1$. If there is $m\geq 2$ such that $f(m)=1$, then we see that $\forall n, f(n)=1$. In the following we assume that $\forall n\geq 2, f(n)>1$. If

$$\exists p>q, x=\ln(f(p))\neq\ln(f(q))=y,$$

since f is non-decreasing, we have $x\geq y$. We choose a rational number $m=\frac{a}{b}$ such that

$$1\leq\frac{x}{y}<m<\frac{\ln(p)}{\ln(q)}.$$

Then we have $q^a<p^b$ and $f(q^a)=e^{ya}>e^{xb}=f(p^b)$ which is contradictory. Thus we conclude that $\forall n, f(n)=n^x$ with $x\geq 0$.

Qusetion 9:

Let Γ be a circle with center O, and $A_1,..,A_n$ be points on Γ. What is the probability that O belongs to the convex hull of $A_1,..,A_n$?

A. We suppose that $n \geq 3$. First let's see the case $n = 3$. We see that O doesn't belong to the triangle if and only if it is obtuse. The greatest angle is $\geq \frac{\pi}{3}$, so the probability of a triangle being obtuse is

$$\frac{\frac{\pi}{2}}{\frac{2\pi}{3}} = \frac{3}{4}.$$

Thus the probability of O belonging to the triangle is $\frac{1}{4}$, which is also the probability of the greatest angle being acute ($\leq \frac{\pi}{2}$).
Now let's go back to the general case. We see that these n points divide 2π into n angles, so the biggest angle is $\geq \frac{2\pi}{n}$. O is not in the convex hull if and only if the biggest angle is $> \pi$.
Therefore the probability that O belongs to the convex hull of $A_1, .., A_n$ is

$$1 - \frac{\pi}{2\pi - \frac{2\pi}{n}} = \frac{n-2}{2n-2}.$$