# A Brief Correction of 2016 Session

*The answer may be wrong or have some errors.*

**Qusetion 1:**
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) \leq f(x) + f(y)$ and $f(x) \leq x$ for all $(x, y) \in \mathbb{R}^2$.

**A.** First we see that $f(0) \leq 2f(0)$. Thus $0 \leq f(0) \leq 0$ and we get $f(0) = 0$. Then we have $0 = f(0) \leq f(x) + f(-x) \leq 0$ thus $f(-x) = -f(x)$. But this means that $f(-x) = -f(x) \leq -x$ so finally we get $f(x) = x$.

**Qusetion 2:**
Find all real numbers $x$ such that $3^x + 4^x = 5^x$.

**A.** We know that $x = 2$ is a solution. Actually it is the only solution and it is easy to see by rewriting the equation in another form:

$$\forall x \in \mathbb{R}^*, \quad (\frac{3}{5})^x + (\frac{4}{5})^x = 1$$

**Qusetion 3:**
Given an $a \in \mathbb{R}_+^*$, let $A_a$ be the set

$$\left\{ \sum_{i=0}^{n} \epsilon_i a^i; \quad n \geq 0, \quad (\epsilon_0, ..., \epsilon_n) \in \{0, 1, -1\}^{n+1} \right\}.$$

(a) For what values of $a$ is the set $A_a$ bounded above ?
(b) Assume $a \geq 2$. Prove that $A_a \cap ]-1, 1[ = \emptyset$.

**A.** If $a \geq 1$, then by taking $(\epsilon_0, ..., \epsilon_n) = (1, ..., 1)$, we see that $A_a$ cannot be bounded. If $a < 1$, it is easy to see that

$$\left| \sum_{i=0}^{n} \epsilon_i a^i \right| \leq \sum_{i=0}^{n} a^i \leq \frac{1}{1-a}.$$

Therefore $A_a$ is bounded.
Now assume that $a \geq 2$. By taking $(\epsilon_0, ..., \epsilon_n) = (0, ..., 0)$, we see that $0 \in A_a$. Since the length of $]-1, 1[$ is 2, we conclude that $A_a \cap ]-1, 1[ = \{0\}$.

**Qusetion 4:**
Let $N_n$ be the number of integers $k \in \{1, ..., n\}$ such that the decimal expansion of $2^k$ terminates with 12. Find the limit of $\frac{N_n}{n}$ as $n$ goes to infinity.

**A.** First we observe that $2^k \equiv 12 \mod 100$ if and only if $2^{k-2} \equiv 3 \mod 25$. Euler's theorem tells us that $2^{20} \equiv 1 \mod 25$ and we verify that 20 is the smallest non-zero positive solution to $2^a \equiv 1 \mod 25$. Now we can conclude that

$$\lim_{n \to \infty} \frac{N_n}{n} = \frac{1}{20}.$$

**Qusetion 5:**
Given a $k \in \mathbb{N}^*$, compute $\int_0^1 (1-t)^k dt$. Use this computation to deduce that, for every $n \in \mathbb{N}^*$,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} = \sum_{k=1}^{n} \frac{1}{k}.$$

**A.** It is easy to see that

$$\int_0^1 (1-t)^k dt = \frac{1}{k+1}.$$

Therefore

$$\begin{aligned} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} &= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \int_0^1 (1-t)^{k-1} dt \\ &= \int_0^1 \frac{1-t^n}{1-t} dt \\ &= \sum_{k=1}^{n} \frac{1}{k}. \end{aligned}$$

**Qusetion 6:**
Consider 100 points in the plane, pairwise distinct. Prove that we can find a straight line that separates the plane in two half-planes, such that each half plane contains exactly 50 points.

**A.** Let's choose an origin, draw two perpendicular axes and represent each point by its coordinate $(x_i, y_i)$. Our goal is to find a unit vector $(\alpha, \beta)$ such that

$$\forall i \neq j, \quad x_i \alpha + y_i \beta \neq x_j \alpha + y_j \beta,$$

which means that each projection on this line is different. If such a unit vector exists, then clearly it is possible to divide 100 points into two disjoint sets of 50 points. It is not difficult to see that such a unit vector exists since there is only a finite amount of unit vectors verifying

$$\exists i \neq j, \quad x_i \alpha + y_i \beta = x_j \alpha + y_j \beta.$$

**Qusetion 7:**
On a line, write down in increasing order all integers between 0 and n (where $n \geq 1$). In a similar way as the construction of the binomial numbers' triangle, write on the second line all the sums of consecutive integers from the first line. Keep going with the third line,... until there is only one integer left. What is this last integer ?

**A.** Let's denote $S_n$ the last integer left. For each $n$, there are $n+1$ lines in the triangle and we can prove by induction that

$$\forall n \geq 1, \quad S_{n+1} = 2S_n + 2^n.$$

Therefore a simple calculation gives us

$$\forall n \geq 1, \quad S_n = n2^{n-1}.$$

**Qusetion 8:**
Let $a, b, c$ be the lengths of the three sides of a triangle with perimeter 1. Prove that

$$\frac{13}{27} \leq a^2 + b^2 + c^2 + 4abc \leq \frac{1}{2}.$$

**A.** Without lost of generality we may assume that $a \geq b \geq c$. Denote

$$f(a,b) = a^2 + b^2 + (1-a-b)^2 + 4ab(1-a-b).$$

We see that

$$\frac{\partial f}{\partial a} = 4a - 2 + 6b - 8ab - 4b^2 = (2a+b-1)(2-4b).$$

Therefore $\frac{\partial f}{\partial a} = 0$ if and only if $a = \frac{1-b}{2}$ or $b = \frac{1}{2}$. The same argument gives us $\frac{\partial f}{\partial b} = 0$ if and only if $b = \frac{1-a}{2}$ or $a = \frac{1}{2}$.
If we have $a \geq \frac{1}{2} \geq b$, then we can see that $f(\frac{a+b}{2}, \frac{a+b}{2}) \leq f(a,b)$ and we have $\frac{1}{2} \geq \frac{a+b}{2}$. If we have $\frac{1}{2} \geq a \geq b$, then $f(a,b)$ is minimal when $(a,b) = (\frac{1}{3}, \frac{1}{3})$ and in this case $f(a,b) = \frac{13}{27}$.
If we have $\frac{1}{2} \geq a \geq b$, then we can see that $f(a,b) \leq f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$. If we have $a \geq \frac{1}{2} \geq b$, we can also find that $f(a,b) \leq f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$.

**Qusetion 9:**
Given $n \in \mathbb{N}^*$, let $D_n$ be the set of divisors of $n$, and $d(n)$ be the number of divisors of $n$. Prove that

$$\forall n \in \mathbb{N}^*, \quad \sum_{k \in D_n} d(k)^3 = \Big(\sum_{k \in D_n} d(k)\Big)^2.$$

**A.** Let's say $n = \prod_i p_i^{a_i}$. Then simply we have,

$$\begin{aligned} \sum_{k \in D_n} d(k)^3 &= \sum_{(b_i)} \prod_i (b_i+1)^3 \\ &= \prod_i \Big(\sum_{k=0}^{a_i} (k+1)^3\Big) \\ &= \Big(\prod_i \frac{(a_i+2)(a_i+1)}{2}\Big)^2 \\ &= \Big(\sum_{k \in D_n} d(k)\Big)^2 \end{aligned}$$

**Qusetion 10:**
We throw infinitely many times a balanced dice with numbers 1 to 6. Let $X_k \in \{1, 2, ..., 6\}$ be the outcome of the k-th throw. For a given $n \in \mathbb{N}^*$, let

$$S_n = \sum_{k=1}^{n} X_k$$

be the sum of the outcomes of the n first throws. For $s \in \mathbb{N}^*$, let $p_s$ be the probability there exists $n$ such that $S_n = s$. Find the minimal value and maximal value of $p_s$ when $s$ assumes all possible values in $\mathbb{N}^*$.

**A.** We can see that

$$p_k = \frac{1}{6}(\frac{7}{6})^{k-1}, \quad k = 1,2,3,4,5,6$$

and

$$\forall k \geq 7, \quad p_k = \frac{1}{6}\sum_{i=1}^{6} p_{k-i}.$$

Then it is easy to see that

$$\min_{n \in \mathbb{N}^*} \{p_n\} = p_1, \quad \max_{n \in \mathbb{N}^*} \{p_n\} = p_6.$$