A Brief Correction of 2017 Session

The answer may be wrong or have some errors.

Qusetion 1:
Find all integers $n \geq 1$ such that $\lfloor \sqrt{n} \rfloor$ divides n.

A. Since $\lfloor \sqrt{n} \rfloor \leq \sqrt{n} < \lfloor \sqrt{n} \rfloor + 1$, one has

$$(\lfloor \sqrt{n} \rfloor)^2 \leq n < (\lfloor \sqrt{n} \rfloor + 1)^2.$$

If $\lfloor \sqrt{n} \rfloor$ divides n, then one must have $n = (\lfloor \sqrt{n} \rfloor)^2$ or $(\lfloor \sqrt{n} \rfloor)(\lfloor \sqrt{n} \rfloor + 1)$ or $(\lfloor \sqrt{n} \rfloor)(\lfloor \sqrt{n} \rfloor + 2)$ Therefore such an integer is k^2 or $k(k+1)$ or $k(k+2)$.

Qusetion 2:
(a) Let A, B, C be three pairwise distinct points in the plane. Find the set of all points M in the plane, such that

$$AC^2 - AM^2 = BC^2 - BM^2$$

(b) Let A, B, C be three pairwise distinct points in the plane; let P, Q, R be three points in the plane. We let D_1 be the straight line perpendicular to BC passing through P; D_2 is the line perpendicular to CA passing through Q; D_3 is the straight line perpendicular to AB passing through R. Prove that D_1, D_2, D_3 all intersect at a single point if, and only if,

$$BP^2 - PC^2 + CQ^2 - QA^2 + AR^2 - RB^2 = 0$$

A. (a) Without loss of generality, one may suppose that $A(a, b), B(c, b), C(0, 0), M(x, y)$. Then one has

$$a^2 + b^2 - (x-a)^2 - (y-b)^2 = c^2 + b^2 - (x-c)^2 - (y-b)^2$$

or

$$(a-c)x = 0$$

The set of points in search is exactly the axis Y.
(b) We note M the intersection of D_1, D_2. Then one has

$$\begin{aligned} BP^2 - PC^2 + CQ^2 - QA^2 &= MB^2 - MA^2 \\ &= RB^2 - RA^2. \end{aligned}$$

Thus M is on D_3 and D_1, D_2, D_3 all intersect at M.
For the other implication, we note M the point of intersection and sum up the three equations defined in (a):

$$\begin{aligned} BP^2 - PC^2 + CQ^2 - QA^2 + AR^2 - RB^2 &= MB^2 - MC^2 + MC^2 - MA^2 + MA^2 - MB^2 \\ &= 0. \end{aligned}$$

Qusetion 3:
Let $n \in \mathbb{N}^*$; let $a_1, ..., a_n$ be elements of $\mathbb{R}^+$, $s = \sum_{i=1}^n a_i$. Prove that

$$\prod_{i=1}^{n}(1+a_i) \leq \sum_{k=0}^{n} \frac{s^k}{k!}$$

A. Since $x \mapsto \ln x$ is convex, one has

$$\frac{1}{n}\sum_{i=1}^{n} \ln(1+a_i) \leq \ln(1+\frac{s}{n}).$$

Therefore

$$\begin{aligned}\prod_{i=1}^{n}(1+a_i) &\leq (1+\frac{s}{n})^n \\ &= \sum_{k=0}^{n}\binom{n}{k}(\frac{s}{n})^k \\ &\leq \sum_{k=0}^{n}\frac{s^k}{k!}\end{aligned}$$

Qusetion 4:
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R}, f(x) \leq x$ and

$$\forall (x, y) \in \mathbb{R}^2, f(x+y) \leq f(x) + f(y)$$

A. By and immediate induction one has

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, f(nx) \leq nf(x)$$

and therefore $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, f(x) = f(x+0) \leq f(x) + nf(0)$, which implies $f(0) = 0$. Thus one gets

$$\forall x \in \mathbb{R}, 0 = f(0) \leq f(x) + f(-x) \leq x + (-x) = 0$$

so one must have $\forall x \in \mathbb{R}, f(x) = x$.

Qusetion 5:
Let ABC be a triangle; let α, β, γ be the internal angles respectively at vertices A, B, C. Prove

$$\cos\alpha\cos\beta\cos\gamma \leq \frac{1}{8}$$

A. The inequality is evident if the triangle is obtuse. If the triangle is acute, since $x \mapsto \cos x$ is convex on $[0, \frac{\pi}{2}]$, one has

$$\begin{aligned}\cos\alpha\cos\beta\cos\gamma &\leq (\frac{\cos\alpha+\cos\beta+\cos\gamma}{3})^3 \\ &\leq (\cos(\frac{\pi}{3}))^3 \\ &= \frac{1}{8}.\end{aligned}$$

Qusetion 6:
Let $n \in \mathbb{N}^*$. Assume that n is not divisible by 2 nor 5. Prove that, in the decimal expansion of n^20, the hundreds digit is even, the tens digit is 0 and the ones digit is 1.

A. Since n can be represented as $10k+m$ where $k \in \mathbb{N}$ and $m = \pm 1, \pm 3$, one has

$$\begin{aligned} n^{20} &= \sum_{i=0}^{20} \binom{20}{i} (10k)^i m^{20-i} \\ &= m^{20} + 20 \cdot (10k) m^{19} + \sum_{i=2}^{20} \binom{20}{i} (10k)^i m^{20-i} \\ &\equiv m^{20} + 20 \cdot (10k) m^{19} \mod 1000. \end{aligned}$$

It is easy to see that $(\pm 1)^{20} \equiv 1 \mod 1000$ and $(\pm 3)^{20} \equiv 9^{10} \equiv 1+100+45 \cdot 100 \equiv 4601 \mod 1000$, so one gets that the hundreds digit is even, the tens digit is 0 and the ones digit is 1.

Qusetion 7:
Let $\alpha \in]0,1[$. Prove that there is no function $f : [0,1] \to \mathbb{R}$ such that

$$\forall (x,y) \in [0,1]^2, (y \geq x \Rightarrow f(y) - f(x) \geq (y-x)^\alpha).$$

A. Suppose that such a function exists, then $f(1) - f(0)$ is well defined. However, since

$$\begin{aligned} \sum_{k=0}^{n-1} f(\frac{k+1}{n}) - f(\frac{k}{n}) &= f(1) - f(0) \\ &\geq \sum_{k=0}^{n-1} (\frac{1}{n})^\alpha \\ &= n^{1-\alpha}, \quad \forall n \in \mathbb{N}^* \end{aligned}$$

one can easily see a contradiction. Therefore, such a function does not exist.

Qusetion 8:
Find the set of $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ not identically zero, such that

$$\forall (x,y) \in \mathbb{R}^2, f(\sqrt{x^2+y^2}) = f(x)f(y)$$

A. First one can see that $\forall x \in \mathbb{R}, f(|x|) = f(x)f(0)$ and $f(0) = f(0)^2$. Thus $f(0) = 1$ and we only need to study for all $x \geq 0$. An immediate induction gives us

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, f(\sqrt{\frac{x^2}{2^n}}) = f(x)^{\frac{1}{2^n}}.$$

Therefore, $\forall x \in \mathbb{R}, f(x) \neq 0$. We define $\forall x \in \mathbb{R}, g(x^2) = \ln(f(\sqrt{x^2}))$ and we have

$$\forall (x,y) \in \mathbb{R}^2, g(x^2+y^2) = g(x^2) + g(y^2).$$

An immediate induction gives us

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, g(nx^2) = ng(x^2) \quad \text{and} \quad g(\frac{x^2}{2^n}) = \frac{g(x^2)}{2^n}.$$

Thus, we can restrict our study on $[0,1]$. Since each number in $[0,1]$ can be represented as a binary number (we talk about density)and we have $g(\frac{1}{2^n}) = \frac{g(1)}{2^n}$, we have $\forall x \in \mathbb{R}, g(x) = g(1)x$ because g is continuous. Finally, we have $f(x) = e^{\alpha x^2}$.

Qusetion 9:
We denote by $|X|$ the number of elements of a finite set X. Let $A_1, .., A_n$ be finite subsets of a set E. For an x in E, denote by $d(x)$ the number of indices $i \in \{1, ..., n\}$ such that $x \in A_i$.

(a) Prove that

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} \sum_{x \in A_i} \frac{1}{d(x)}$$

(b) Then deduce that

$$|\bigcup_{i=1}^{n} A_i| \geq \sum_{i=1}^{n} \frac{|A_i|^2}{\sum_{j=1}^{n} |A_i \cap A_j|}$$

A. (a)

$$\begin{aligned}\sum_{i=1}^{n} \sum_{x \in A_i} \frac{1}{d(x)} &= \sum_{x \in (\bigcup_{i=1}^{n} A_i)} \sum_{i=1}^{n} \frac{1}{d(x)} \\ &= \sum_{x \in (\bigcup_{i=1}^{n} A_i)} 1 \\ &= |\bigcup_{i=1}^{n} A_i|.\end{aligned}$$

(b) We define $\forall (i,j) \in \{1, ..., n\}^2$, $I_{i,j}(x) = 1$ if $x \in A_i \cap A_j$ and 0 if not. Then on can see that $\forall x \in A_i, d(x) = \sum_{j=1}^{n} I_{i,j}(x)$ and

$$\begin{aligned}\sum_{x \in A_i} d(x) &= \sum_{x \in A_i} \sum_{j=1}^{n} I_{i,j}(x) \\ &= \sum_{j=1}^{n} \sum_{x \in A_i} I_{i,j}(x) \\ &= \sum_{j=1}^{n} |A_i \cap A_j|.\end{aligned}$$

By Cauchy inequality, one has

$$\sum_{j=1}^{n} |A_i \cap A_j| \sum_{x \in A_i} \frac{1}{d(x)} \geq |A_i|^2.$$

Therefore,

$$|\bigcup_{i=1}^{n} A_i| \geq \sum_{i=1}^{n} \frac{|A_i|^2}{\sum_{j=1}^{n} |A_i \cap A_j|}.$$

Qusetion 10:

Let E be a finite set, $n \geq 2$ an integer; let $A_1, ..., A_n$ and $B_1, ..., B_n$ be subsets of E. Assume for every i in $\{1, .., n\}$, $A_i \cap B_i = \emptyset$ and if i and j are distinct elements of $\{1, .., n\}$, then $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$. Given a $p \in [0, 1]$, prove that

$$\sum_{i=1}^{n} p^{|A_i|}(1-p)^{|B_i|} \leq 1.$$

[One could introduce Bernoulli-type random variables.]

A. Let $E = \{1, 2, ..., k\}$. We have k independent coin such that each time we throw it, the probability of having the face is p and that of having the back is $(1-p)$. For $j = 1, ..., n$, we denote by W_j the event "for all i in A_j, the i^{th} coin is face and for all i in B_j, the i^{th} coin is back". Therefore the probability of having W_j is $p^{|A_j|}(1-p)^{|B_j|}$. In addition, if $i \neq j$, the probability of having at the same time W_i and W_j is 0 since $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ and a coin cannot be at the same time face and back. Thus, we have

$$\sum_{i=1}^{n} p^{|A_i|}(1-p)^{|B_i|} \leq \text{the sum of the probability of each possible event} = 1.$$