MPSI Class Entrance Test 2004

Test time : 4 hours
English Version

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The following exercises can be solved independently and done in any order. The question are listed from easiest to most difficult. Calculators are not permitted. Solutions should be written in French.

1. Find all integers $x \in \mathbb{N}$ such that $\dfrac{x^3+3}{x+3}$ is an integer.
2. Prove that the sequence
$$\left(\left(1+\frac{1}{n}\right)^n\right)_{n\geq 1}$$
is increasing.
3. For all real number p we consider the line $D_p : y = px + p^2$. Find, and draw, the set of all points in the plane that are not on any of the D_p lines.
4. Show that the symetrical points of the orthocenter of a triangle, in relation to its sides are on the circumcircle of the triangle.
5. Find two real numbers a et b such that
$$\int_0^\pi (at+bt^2)\cos(nt)\mathrm{d}t = \frac{1}{n^2}$$
for all $n \geq 1$.
6. a. The squares of a $1 \times n$ chessboard must be colored with d colors but in such a way that two adjacent squares have different colors. Find the number of ways this can be done.
b. Solve the same question for a $2 \times n$ chessboard (2 rows and n columns).
7. On the first line we write the integers from 0 to n ($n \geq 1$). Like in the Pascal triangle, we write on the second line the sums of two consecutives integers from the first line, and so on. What is the integer we get at the end on the last line ?
$$\begin{array}{ccccccccccccc}
0 & & 1 & & 2 & & 3 & \cdots & & n-1 & & n \\
& 1 & & 3 & & 5 & & \cdots & & & 2n-1 & \\
& & 4 & & 8 & & & \cdots & & \cdot^{\cdot^{\cdot}} & & \\
& & & \ddots & & & & \cdot^{\cdot^{\cdot}} & & & &
\end{array}$$
8. If n is an integer let $f(n)$ be the sum of the square of the decimal digits of n. For example, if $n = 37$ we have $f(n) = 3^2 + 7^2 = 58$.
A integer a being choosen, we study the sequence $(u_n)_{n\geq 0}$ defined by $u_0 = a$ and the recurrence relation $u_{n+1} = f(u_n)$. Prove that this sequence is eventually periodic (this means that we can find $T \geq 1$ and $N \in \mathbb{N}$ such that $u_{n+T} = u_n$ for $n \geq N$).

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