

MPSI Class Entrance Test 2009

Test time : 4 hours
English Version

The following exercises can be solved independently and done in any order. The questions are listed from easiest to most difficult. Calculators are not permitted. Solutions should be written in French or in English.

1. Find all real numbers $x > 0$ such that $x^{\sqrt{x}} = \sqrt{x^x}$.
2. If p is a real number let D_p be the line $y = px + p(p - 1)$ in the plane \mathbb{R}^2 . Find and draw the set of points (x, y) which are not on any line D_p .
3. In the plane, let D be a line, Γ and Γ' two circles, r the radius of Γ , r' the radius of Γ' . Suppose that Γ and Γ' are on the same side of D , that Γ and Γ' are tangent to D respectively at the point M and at the point M' and that Γ is tangent to Γ' . Express MM' as a function of r and r' .
4. Find the greatest real number k such that $x^2 + xy + y^2 \geq k(x^2 + y^2)$ for any pair (x, y) of real numbers.
5. Let $m \geq 1$ be an integer. Find the greatest integer $n \geq 1$ such that 2^n divides $5^{2^m} - 1$.
6. Let $k \geq 2$ be an integer and E be the set :

$$\{n^k, n \in \mathbb{N}^*\} = \{1, 2^k, 3^k, 4^k, \dots\}.$$

Prove that E does not contain any arithmetic progression.

7. Let Γ be a semicircle of center O and radius 1. If $2n + 1$ points $A_1, A_2, \dots, A_{2n+1}$ are given on Γ , prove that :

$$\left\| \overrightarrow{OA_1} + \overrightarrow{OA_2} + \dots + \overrightarrow{OA_{2n+1}} \right\| \geq 1.$$

8. Let $n \geq 1$ be an integer. A box contains n tokens labelled from 1 to n . A random set of tokens (possibly empty) is drawn from the box. They are put back into it and a second drawing is performed. What is the probability that the second set of tokens contains the first one?
